

REGULARITY AND STOCHASTIC HOMOGENIZATION OF FULLY NONLINEAR EQUATIONS WITHOUT UNIFORM ELLIPTICITY

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ABSTRACT. We prove regularity and stochastic homogenization results for certain degenerate elliptic equations in nondivergence form. The equation is required to be strictly elliptic, but the ellipticity may oscillate on the microscopic scale and is only assumed to have a finite d th moment, where d is the dimension. In the general stationary-ergodic framework, we show that the equation homogenizes to a deterministic, uniformly elliptic equation, and we obtain an explicit estimate of the effective ellipticity which is new even in the uniformly elliptic context. Showing that such an equation behaves like a uniformly elliptic equation requires a novel reworking of the regularity theory. We prove deterministic estimates depending on averaged quantities involving the distribution of the ellipticity, which are controlled in the macroscopic limit by the ergodic theorem. We show that the moment condition is sharp by giving an explicit example of an equation whose ellipticity has a finite p th moment, for every $p < d$, but for which regularity and homogenization break down. While these results are new even for linear equations, we prove them in the fully nonlinear context. In probabilistic terms, our homogenization results correspond to quenched invariance principles for controlled diffusion processes in random media, including linear diffusions as well as diffusions controlled by one controller or two competing players.

1. INTRODUCTION

We prove stochastic homogenization and regularity estimates for fully nonlinear elliptic equations in nondivergence form without the assumption of uniform ellipticity. The equations we consider are strictly elliptic but may have ellipticities which are arbitrarily large and oscillating on the microscopic scale. We derive new (deterministic) regularity estimates and show that, under the assumption that the d th moment of the ellipticity is finite, such a degenerate equation homogenizes in the macroscopic limit to an effective equation which is uniformly elliptic. Our analysis yields an explicit estimate for the effective ellipticity which is new, to our knowledge, even in the linear, uniformly elliptic setting. In terms of probability, the main homogenization result, in the linear case, is equivalent to a quenched invariance principle for a diffusion in a random environment. For a nonlinear, positively homogeneous equation, the homogenization result gives similar information about the quenched behavior of controlled diffusions in random environments.

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The simplest example of interest is the linear equation

$$(1.1) \quad - \sum_{i,j=1}^d a_{ij} \left(\frac{x}{\varepsilon}, \omega \right) u_{x_i x_j}^\varepsilon = f \quad \text{in } U \subseteq \mathbb{R}^d, \quad d \geq 1.$$

The coefficient matrix (a_{ij}) , which depends on the random parameter ω , called *the environment*, is assumed to be stationary-ergodic and to satisfy the ellipticity condition

$$(1.2) \quad \lambda(\omega) |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(0, \omega) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d,$$

where $\Lambda > 0$ is a given constant and $\lambda = \lambda(\omega)$ is a nonnegative random variable. Papanicolaou and Varadhan [25, 26] proved that, if the equation is uniformly elliptic, i.e., $\lambda(\omega) \geq \lambda_0 > 0$, then in the almost sure asymptotic limit $\varepsilon \rightarrow 0$ the equation (1.1) is governed by an effective equation of the form

$$- \sum_{i,j=1}^d \bar{a}_{ij} u_{x_i x_j} = f,$$

where the coefficient matrix (\bar{a}_{ij}) is uniformly elliptic with ellipticity Λ/λ_0 , that is, for every $\xi \in \mathbb{R}^d$,

$$(1.3) \quad \lambda_0 |\xi|^2 \leq \sum_{i,j=1}^d \bar{a}_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2.$$

In this paper we extend this result to the case that $\lambda > 0$ and $\inf \lambda = 0$ under the assumption that $\mathbb{E}[\lambda^{-d}] < \infty$. Due to the presence of small pockets of arbitrarily large ellipticity which become dense for small ε , it is by no means clear at first glance that (1.1) should behave, in the macroscopic limit, like a uniformly elliptic equation. Our result demonstrates that this finite moment condition on λ^{-1} ensures that these tiny regions of very large ellipticity may be effectively controlled and that (1.1) becomes a uniformly elliptic equation in the limit $\varepsilon \rightarrow 0$. Furthermore, we show that the moment condition is sharp by exhibiting an explicit example in arbitrary dimension $d \geq 1$ of a linear equation, in a random environment with a finite range of dependence, for which homogenization breaks down but the ellipticity has finite p th moment for every $p < d$.

Previously, in this linear setting, a result similar to our homogenization result has been proved in a recent paper of Guo and Zeitouni [19] using probabilistic methods. They give a quenched invariance principle for random walks in a random environment, which has an equivalent formulation in terms of stochastic homogenization of a discrete equation (on the lattice \mathbb{Z}^d rather than \mathbb{R}^d). They require a slightly stronger moment condition, namely that λ^{-1} have a finite p th moment for some $p > d$. That homogenization occurs in the critical case $p = d$ is new here, as well as the fact that it fails for every $p < d$.

While our results are therefore of interest in the linear setting, we analyze much more general fully nonlinear equations of the form

$$(1.4) \quad F \left(D^2 u^\varepsilon, \frac{x}{\varepsilon}, \omega \right) = 0.$$

Such nonlinear equations are more difficult to analyze than (1.1), due to the absence of invariant measures—the key tool used in [25, 26] (and [19]) to overcome the problem of the “lack of compactness.” The results here are the first for such equations without a uniform ellipticity assumption. Previously, the homogenization of fully nonlinear equations in stationary-ergodic media was proved in the uniformly elliptic case by Caffarelli, Souganidis and Wang [8]. They introduced a new method based on the obstacle problem, a strategy which we also use in this paper.

The problem one encounters when trying to homogenize (1.4) outside of the uniformly elliptic regime is that most of the regularity theory needed to implement the method of [8] is destroyed by (even tiny) regions of high ellipticity. It therefore seems hopeless, at first glance, to implement the techniques of [8], since they make heavy use of the regularity tools.

To overcome this difficulty prove new (deterministic) regularity estimates in which the dependence on a uniform upper bound for the ellipticity is replaced by that of its L^d -norm. In particular, we prove a decay of oscillation lemma at unit scale. This result, and the new arguments we introduce to obtain it, are of independent interest. Indeed, in sharp contrast to the situation for divergence form equations, there are very few results in the literature for equations in nondivergence form which provide estimates of solutions in terms of averaged quantities.

The estimates refine the classical regularity theory [6] and require several new ideas. One of the basic techniques involves using the area formula to estimate the size of certain “contact sets” between supersolutions and certain families of smooth test functions. This method is a generalization of the the classical ABP inequality and was previously used by Savin in his proof of De Giorgi’s conjecture [28] and his beautiful proof of the Harnack inequality in [27]. In those works, supersolutions are touched from below not only by planes, but by translations of balls and paraboloids. One of the key arguments in the present paper involves touching from below with translations of the *singular function* $|x|^{-\alpha}$, for suitably large $\alpha > 0$.

Since the L^d norm of the ellipticity is controlled on the macroscopic scale almost surely by the ergodic theorem, the regularity results provide effective control on the solutions of (1.4) for small ε . This allows us to homogenize the equation by suitably adapting the arguments of [8]. We expect this two-step approach to homogenization, in which one obtains “effective” regularity and then uses this to homogenize the equation, to be useful in other situations.

As a byproduct of our analysis, we obtain an estimate for the effective ellipticity which is new even in the uniformly elliptic case, which states that (1.3) holds for a $\lambda_0 > 0$ which, in addition to Λ and d , depends only on $\mathbb{E}[\lambda^{-d}]$. It is, to our knowledge, the first such bound for the homogenized coefficients of nondivergence form equations which is nontrivial in the sense that it is given in terms of on the averaged microscopic behavior of the equation rather than its uniform properties.

As mentioned above, the homogenization result has an equivalent probabilistic formulation, at least in the linear case, as a quenched invariance principle for the corresponding diffusion in the random environment. It also provides information regarding the recurrence or transience of the diffusion (see [19]). If F is nonlinear but positively homogeneous, the fully nonlinear equation (1.4) is a Bellman-Isaacs equation which arises in the theory of stochastic optimal control and two-player

stochastic differential games, and the homogenization result yields similar information about these more general diffusion processes. Although we do not explore this point here, we remark that the recurrence versus transience of such controlled diffusion processes in an isotropic environment was characterized in [3], and this result applied to the effective operator \overline{F} , together with its proof, gives information about the corresponding questions for controlled diffusions in random environments.

We now give the precise statement of our results, beginning with the modeling assumptions.

The model. We work in Euclidean space \mathbb{R}^d in dimension $d \geq 1$. The random environment is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with an ergodic group $\tau = (\tau_y)_{y \in \mathbb{R}^d}$ of \mathcal{F} -measurable, \mathbb{P} -preserving transformations on Ω . That is, the action τ of \mathbb{R}^d on Ω satisfies

$$(1.5) \quad \mathbb{P}[A] = \mathbb{P}[\tau_y A] \quad \text{for every } y \in \mathbb{R}^d, A \in \mathcal{F}$$

and, for every $A \in \mathcal{F}$,

$$(1.6) \quad \tau_y A = A \quad \text{for every } y \in \mathbb{R}^d \quad \text{implies that} \quad \mathbb{P}[A] = 0 \quad \text{or} \quad \mathbb{P}[A] = 1.$$

The nonlinear elliptic operator is a map $F : \mathbb{S}^d \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ (here \mathbb{S}^d denotes the space of d -by- d symmetric matrices) which satisfies each of the following four conditions:

(F1) *Stationarity:* for every $M \in \mathbb{S}^d$, $y, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$F(M, y, \tau_z \omega) = F(M, y + z, \omega).$$

(F2) *Local uniform ellipticity:* there exists a constant $\Lambda \geq 1$ and a nonnegative random variable $\lambda : \Omega \rightarrow [0, \Lambda]$ such that $\mathbb{P}[\lambda > 0] = 1$ and, for every $M, N \in \mathbb{S}^d$, $\omega \in \Omega$ and $y \in B_1$,

$$\mathcal{P}_{\lambda(\omega), \Lambda}^-(M - N) \leq F(M, y, \omega) - F(N, y, \omega) \leq \mathcal{P}_{\lambda(\omega), \Lambda}^+(M - N).$$

(Here \mathcal{P}^\pm are the usual Pucci extremal operators; see the next section).

(F3) *Uniform continuity and boundedness:* for each $R > 0$,

$$\{F(\cdot, \cdot, \omega) : \omega \in \Omega\} \quad \text{is uniformly equicontinuous on } B_R \times \mathbb{R}^d$$

and

$$\operatorname{ess\,sup}_{\omega \in \Omega} |F(0, 0, \omega)| < +\infty.$$

(F4) *Bounded moment of the ellipticity:* the random variable λ satisfies

$$\mathbb{E} [\lambda^{-d}] < +\infty.$$

The main result. We now present the homogenization result, which for simplicity we state in terms of the Dirichlet problem

$$(1.7) \quad \begin{cases} F\left(D^2 u^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0 & \text{in } U, \\ u^\varepsilon = g & \text{on } \partial U. \end{cases}$$

Here $U \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain and $g \in C(\partial U)$, and the PDE is to be understood in the viscosity sense (c.f. [10, 6]). By modifying our argument in a

very minor way (only small changes in part three of Section 4), we may homogenize any other well-posed problem involving F , including parabolic equations like

$$u_t + F\left(D^2u, \frac{x}{\varepsilon}, \omega\right) = 0$$

with appropriate boundary/initial conditions.

Note that by (F1) and (F2), for each $\varepsilon > 0$, the equation (1.7) is uniformly elliptic with probability one. Indeed, if we take $\{B(x_j, 1) : 1 \leq j \leq k\}$ to be a finite covering of $\varepsilon^{-1}U$, then its ellipticity is bounded by the random variable

$$\Lambda \sup_{1 \leq j \leq k} \lambda^{-1}(\tau_{x_j/\varepsilon}\omega),$$

which is almost surely finite by (F2). See (2.1) below. As a consequence, (1.7) is well-posed and has a unique viscosity solution $u^\varepsilon = u^\varepsilon(x, \omega)$ belonging to $C(\overline{U})$.

The main homogenization result is:

Theorem 1. *Assume (F1), (F2), (F3) and (F4). Then there exists an event $\Omega_1 \in \mathcal{F}$ of full probability, a positive constant $0 < \lambda_0 < \Lambda$ which depends only on d , Λ and $\mathbb{E}[\lambda^{-d}]$ and a function $\overline{F} : \mathbb{S}^d \rightarrow \mathbb{R}$ which satisfies*

$$\mathcal{P}_{\lambda_0, \Lambda}^-(M - N) \leq \overline{F}(M) - \overline{F}(N) \leq \mathcal{P}_{\lambda_0, \Lambda}^+(M - N)$$

such that, for every $\omega \in \Omega_1$, every bounded Lipschitz domain $U \subseteq \mathbb{R}^d$ and each $g \in C(\partial U)$, the unique solution $u^\varepsilon = u^\varepsilon(x, \omega)$ of the boundary value problem (1.7) satisfies

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in U} |u^\varepsilon(x, \omega) - u(x)| = 0,$$

where $u \in C(\overline{U})$ is the unique solution of the Dirichlet problem

$$(1.8) \quad \begin{cases} \overline{F}(D^2u) = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

A brief literature review. The homogenization of linear uniformly elliptic equations in random media originated in the work of Papanicolaou and Varadhan [25, 26] and Kozlov [23, 24]. Later, Dal Maso and Modica [11, 12] obtained stochastic homogenization results for nonlinear equations in divergence form and convex variational problems. The homogenization of uniformly elliptic, nonlinear equations in nondivergence form was first considered in the periodic setting by Evans [15] and much later by Caffarelli, Souganidis and Wang [8] in random media. In contrast to the divergence form case (c.f. [9]), little seems to be known about the homogenized coefficients for nondivergence form equations, even in the periodic case, other than what is inherited from the uniform properties of the medium. As far as quantitative homogenization results, we mention the work of Yurinskii [29] and Gloria and Otto [17, 18] for linear equations and Caffarelli and Souganidis [7] for fully nonlinear equations.

The modern regularity theory for elliptic equations in nondivergence form began in the 1980s with the groundbreaking work of Krylov and Safonov, Evans, Caffarelli and others, and we refer to [6] and the references there for more. Recently, there has been some progress in obtaining Harnack inequalities and Hölder regularity for certain nonlinear degenerate equations (see for example [20, 14, 21]). In these

works, the degeneracy of the equation is typically compensated in some way by dependence on the gradient. A typical model equation considered is

$$(1.9) \quad |Du|^\gamma F(D^2u) = 0,$$

where $\gamma > 0$ and F is uniformly elliptic. A solution u of (1.9) may only be irregular if $|Du|$ is small, and this allows to compensate for the degeneracy. This is a very different situation from the “naked” degeneracy of the equations considered here.

Outline of the paper. In the next section we give some preliminary results and notation needed later in the paper and make some comments about our assumptions. In Section 3 we develop the deterministic regularity theory. The proof of Theorem 1 is then given in Section 4. Finally, in Section 5 we construct an explicit example to show that the moment condition (F4) is sharp.

2. PRELIMINARIES

In this section we present some background results needed in the rest of the paper, including the statements of the ergodic theorems we cite, some remarks about our model and some general remarks concerning viscosity solutions and semiconcave functions. We begin by reviewing the notation.

Notation. The symbols C and c denote positive constants which may vary at each occurrence and which typically depend on known quantities. We work in Euclidean space \mathbb{R}^d for $d \geq 1$. We denote the set of natural numbers by $\mathbb{N} := \{0, 1, \dots\}$ and \mathbb{Q} is the set of rational numbers. The family of bounded Lipschitz subsets of \mathbb{R}^d is denoted by \mathcal{L} . The open ball centered at $y \in \mathbb{R}^d$ with radius $r > 0$ is $B_r(y) := \{x \in \mathbb{R}^d : |x - y| < r\}$ and we write $B_r := B_r(0)$. If $E \subseteq \mathbb{R}^d$ is a bounded Borel set, then \overline{E} is its closure and $|E|$ is the Lebesgue measure of E . If $f \in L^1(E)$, then the average of f in E is $\bar{f}_E f(x) dx := |E|^{-1} \int_E f(x) dx$. If $f : E \rightarrow \mathbb{R}$, then we denote $\text{osc}_E f := \sup_E f - \inf_E f$. The characteristic function of a Borel set E is χ_E . We work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as described in the previous section. The indicator random variable of an event $A \in \mathcal{F}$ is written $\mathbf{1}_A$. We say that $A \in \mathcal{F}$ is of *full probability* if $\mathbb{P}[A] = 1$. The space of symmetric d -by- d matrices is \mathbb{S}^d . If $M, N \in \mathbb{S}^d$, we write $M \geq N$ if the eigenvalues of $M - N$ are nonnegative. If $x, y \in \mathbb{R}^d$, then $x \otimes y$ denotes the d -by- d matrix with entries $(x_i y_j)$. The trace of $M \in \mathbb{S}^d$ is $\text{tr}(M)$. Recall that any $M \in \mathbb{S}^d$ can be uniquely expressed as a difference $M = M_+ - M_-$ where $M_+ M_- = 0$ and $M_+, M_- \geq 0$. In particular, if $r \in \mathbb{R}$, then we write $r_+ := \max\{0, r\}$ and $r_- := (-r)_+$. The Pucci extremal operators \mathcal{P}^\pm are defined for $0 < \mu \leq \Lambda$ and $M \in \mathbb{S}^d$ by

$$\mathcal{P}_{\mu, \Lambda}^+(M) := -\mu \text{tr}(M_+) + \Lambda \text{tr}(M_-) \quad \text{and} \quad \mathcal{P}_{\mu, \Lambda}^-(M) := -\Lambda \text{tr}(M_+) + \mu \text{tr}(M_-).$$

The elementary properties of the Pucci operators can be found in [6]. Here we remark only that they are uniformly elliptic, $\mathcal{P}_{\mu, \Lambda}^+$ is convex and $\mathcal{P}_{\mu, \Lambda}^-$ is concave.

Brief remarks concerning the assumptions. Note that the restriction of (F2) to $y \in B_1$ is merely for convenience, it may be extended to all $y \in \mathbb{R}^d$ by stationarity. Indeed, the combination of (F1) and (F2) yields, for all $y, z \in \mathbb{R}^d$ with $|y - z| < 1$,

$$(2.1) \quad \mathcal{P}_{\lambda(\tau_z \omega), \Lambda}^-(M - N) \leq F(M, y, \omega) - F(N, y, \omega) \leq \mathcal{P}_{\lambda(\tau_z \omega), \Lambda}^+(M - N).$$

In light of (2.1), it is convenient to abuse notation by writing $\lambda(z, \omega) = \lambda(\tau_z \omega)$. Note also that due to (F3) we may suppose that

$$(2.2) \quad \{\lambda(\cdot, \omega) : \omega \in \Omega\} \text{ is uniformly equicontinuous on } \mathbb{R}^d.$$

Otherwise we simply redefine λ to be the largest quantity which satisfies (F2), which then satisfies (2.2) by (F3). The operators on the leftmost and rightmost side of (2.1) are the minimal and maximal operators, respectively, which satisfy conditions (F1)–(F3). In particular, since $\lambda(\cdot, \omega) > 0$, our equation is locally uniformly elliptic in the sense that, almost surely, $\inf_V \lambda(\cdot, \omega) > 0$ for each $V \in \mathcal{L}$.

Using ergodicity, we may improve the second part of (F3) to

$$\sup_{y \in \mathbb{R}^d} \operatorname{ess\,sup}_{\omega \in \Omega} |F(0, y, \omega)| < +\infty.$$

Using then the continuity of F and intersecting the event on which the latter holds over all the rational points of \mathbb{R}^d , we obtain

$$\operatorname{ess\,sup}_{\omega \in \Omega} \sup_{y \in \mathbb{R}^d} |F(0, y, \omega)| < +\infty.$$

Applying also (F2), this yields, for $C_0 := \operatorname{ess\,sup}_{\omega \in \Omega} |F(0, y, \omega)|$ and all $M \in \mathbb{S}^d$,

$$(2.3) \quad \operatorname{ess\,sup}_{\omega \in \Omega} \sup_{y \in \mathbb{R}^d} |F(M, y, \omega)| \leq C_0 + \Lambda \operatorname{tr}(M_+ + M_-) \leq C(1 + |M|).$$

A brief remark concerning viscosity solutions. All differential inequalities in this paper are to be interpreted in the viscosity sense (c.f. [6, 10]). We remark that, while it is not obvious– in fact, it is *equivalent* to the comparison principle– we have transitivity of inequalities in the viscosity sense (see [2, Lemma 3.2]). For example, if $V \in \mathcal{L}$ and $u, v \in C(V)$ satisfy

$$F(D^2u, y, \omega) \geq 0 \quad \text{and} \quad F(D^2v, y, \omega) \leq 0 \quad \text{in } V$$

then formally it follows that for $w := u - v$ we have

$$(2.4) \quad 0 \leq F(D^2u, y, \omega) - F(D^2v, y, \omega) \leq \mathcal{P}_{\lambda(x, \omega), \Lambda}^+(D^2w).$$

We emphasize that we may also deduce $\mathcal{P}_{\lambda(x, \omega), \Lambda}^+(D^2w) \geq 0$ in the viscosity sense, and make other similar formal deductions rigorous, using [2, Lemma 3.2].

Pointwise notions of twice differentiability and $C^{1,1}$. We require the following pointwise regularity notions. We say that $u \in C(B(0, 1))$ is *twice differentiable* at $x \in B_1$ if there exist $(X, p) \in \mathbb{S}^d \times \mathbb{R}^d$ such that

$$\limsup_{r \rightarrow 0} \sup_{y \in B_r(x)} r^{-2} |u(y) - u(x) - p \cdot (y - x) - \frac{1}{2}(y - x) \cdot X(y - x)| = 0,$$

in which case we write $D^2u(x) := X$ and $Du(x) := p$. We also say that u is $C^{1,1}$ on a set $E \subseteq B(0, 1)$ if u is differentiable at each point of E and

$$\sup_{x \in E} \sup_{y \in B_1} \frac{|u(y) - u(x) - Du(x) \cdot (y - x)|}{|x - y|^2} < +\infty.$$

A function u is *semiconcave* if there exist $k > 0$ such that the map $x \mapsto u(x) - k|x|^2$ is concave. In this paper we rely many times on the observation that, for any $a > 0$, a semiconcave function is $C^{1,1}$ on the set of points at which it can be touched from below by a C^2 functions with Hessian bounded by a . Moreover, by Rademacher's

theorem and the Lebesgue differentiation theorem, any $C^{1,1}$ function on a set E is twice differentiable at (Lebesgue) almost every point of E .

Infimal convolution. We recall a standard tool (c.f. [10, 6] for details) in the theory of viscosity solutions. If $u \in C(B_1)$ then its infimal convolution is denoted by

$$u_\varepsilon(x) := \inf_{y \in B_1} \left(u(y) + \frac{2}{\varepsilon} |x - y|^2 \right).$$

The function u_ε is more regular than u and, in particular, is semiconcave. It is a good approximation to u in the sense that $u_\varepsilon \rightarrow u$ locally uniformly in B_1 as $\varepsilon \rightarrow 0$. Moreover, if $f, \lambda \in C(B_1)$, $\lambda > 0$ and

$$\mathcal{P}_{\lambda(x), \Lambda}^+(D^2 u) \geq f \quad \text{in } B_1,$$

then there exist sequences of functions $\lambda'_\varepsilon, f'_\varepsilon \in C(B_1)$ which converge locally uniformly to λ and f , respectively, as $\varepsilon \rightarrow 0$, such that u_ε satisfies

$$\mathcal{P}_{\lambda'_\varepsilon(x), \Lambda}^+(D^2 u_\varepsilon) \geq f'_\varepsilon \quad \text{in } B_{1-r_\varepsilon}$$

where $r_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. We refer to [6, 10] for details. For us, the principle utility of these approximations is the semiconcavity of u_ε . If u_ε can be touched from below by a smooth function φ at some point $z \in B_1$, then u_ε is $C^{1,1}$ at z , with norm depending only ε and $|D^2 \varphi(z)|$. See [6, Theorem 5.1].

Statements of the ergodic theorems. We next recall the two versions of the (multiparameter) ergodic theorem used in this paper. The first is nearly a consequence of the second, but since it is simpler we give it separately. A nice proof can be found in Becker [4].

We emphasize that the assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in particular (1.5) and (1.6), are in force. Recall that \mathcal{L} denotes the set of all bounded Lipschitz subsets of \mathbb{R}^d .

Proposition 2.1 (Wiener's ergodic theorem). *Let $f \in L^1(\Omega)$. Then there exists a subset $\Omega_0 \in \mathcal{F}$ of full probability such that, for every $\omega \in \Omega_0$ and $V \in \mathcal{L}$,*

$$(2.5) \quad \lim_{t \rightarrow \infty} \int_{tV} f(\tau_y \omega) dy = \mathbb{E}[f].$$

In particular, the map $y \mapsto f(\tau_y \omega)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$ for every $\omega \in \Omega_0$.

The version of Proposition 2.1 proved in [4] actually requires V to be star-shaped with respect to the origin. As is well-known, this restriction may be removed as follows. First we notice that the conclusion holds for any cube $V = Q$ with sides parallel to the coordinate axes, since any such cube either contains the origin or has the property that, for some larger cube \tilde{Q} , both $\tilde{Q} \setminus Q$ and \tilde{Q} are star-shaped with respect to the origin. Since it holds for such cubes, it holds for an arbitrary finite disjoint union of them and hence any $V \in \mathcal{L}$ by approximation.

We next state the multiparameter subadditive ergodic theorem of Akcoglu and Krengel [1] as modified by Dal Maso and Modica [12], which requires some further

notation. We denote by \mathcal{U}_0 the family of bounded subsets of \mathbb{R}^d . A function $f : \mathcal{U}_0 \rightarrow \mathbb{R}$ is *subadditive* if,

$$f(A) \leq \sum_{j=1}^k f(A_j)$$

whenever $k \in \mathbb{N}$ and $A, A_1, \dots, A_k \in \mathcal{U}_0$ are such that $\cup_{j=1}^k A_j \subseteq A$, the sets A_1, \dots, A_k are pairwise disjoint, and $|A \setminus \cup_{j=1}^k A_j| = 0$. Let \mathcal{M} be the collection of subadditive functions $f : \mathcal{U}_0 \rightarrow \mathbb{R}$ which satisfy

$$0 \leq f(A) \leq |A| \quad \text{for every } A \in \mathcal{U}_0.$$

A *subadditive process* is a function $f : \Omega \rightarrow \mathcal{M}$. It is sometimes convenient to write $f(A, \omega) = f(\omega)(A)$, in which case we have $f(A, \tau_y \omega) = f(y + A, \omega)$.

Proposition 2.2 (Subadditive ergodic theorem). *Let $f : \Omega \rightarrow \mathcal{M}$ be a subadditive process. Then there exists an event $\Omega_0 \in \mathcal{F}$ of full probability and a constant $0 \leq a \leq 1$ such that, for every $\omega \in \Omega_0$ and $V \in \mathcal{L}$,*

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{f(tV, \omega)}{|tV|} = a.$$

This version of the subadditive ergodic theorem is [1, Proposition 1] in the special case that \mathcal{L} is replaced by the family of all cubes, and we recover the general case by an easy approximation argument.

3. REGULARITY IN THE MACROSCOPIC LIMIT

The classical regularity theory (as developed for example in [6]) does not directly help us to homogenize (1.4) because, as ε becomes small, the ergodic theorem guarantees that the set where (1.4) has very high ellipticity becomes dense. What we need are estimates which do not degenerate as $\varepsilon \rightarrow 0$, and for this it is necessary to revisit the regularity from the beginning.

What the ergodic theorem ensures is that, almost surely in ω , for every $\mu > 0$,

$$(3.1) \quad \lim_{\varepsilon \rightarrow 0} \int_{V \cap \{\lambda < \mu\}} \lambda^{-d} \left(\frac{x}{\varepsilon}, \omega \right) dx = \mathbb{E}[\lambda^{-d} \mathbf{1}_{\{\lambda < \mu\}}].$$

In this section we develop a deterministic regularity theory for solutions of (1.4) which will be robust in the almost sure macroscopic limit $\varepsilon \rightarrow 0$ by virtue of (3.1).

Since the random environment plays no role here, we drop dependence on ω . Throughout this section we consider a continuous function $\lambda : \mathbb{R}^d \rightarrow (0, \Lambda]$, and we study the regularity of subsolutions and/or supersolutions of the extremal operators $\mathcal{P}_{\lambda(x), \Lambda}^\pm$ in bounded Lipschitz domains $V \in \mathcal{L}$. Our estimates must depend only on d, Λ and, for $\mu > 0$, the quantities

$$\int_{V \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx.$$

As it is purely deterministic, the regularity developed here is of independent interest.

The primary goal is to obtain an improvement of oscillation result on unit scales, giving us a modulus of continuity (and a Hölder estimate in the macroscopic limit for solutions of (1.4)). Our arguments are loosely based on the arguments in the classical regularity theory [6], with some nice modifications due to Savin [27], but

require several new ideas to overcome the degeneracy of the equation. For instance, “two important tools” are introduced in [6, Section 4.1] which are used repeatedly in what has become the standard proof of the Harnack inequality. In our situation, neither of these tools can be applied in a straightforward manner.

First, showing that an appropriate barrier (or “bump”) function exists— which is easy in the uniformly elliptic situation (see [6, Lemma 4.1])— is a very nontrivial matter. Here we construct a barrier by touching a candidate function from below by translations of the singular function $|x|^{-\alpha}$ with $\alpha \gg 1$ and then adapting the proof of the ABP inequality to show that the corresponding contact set would be too large if the function failed to be a barrier. Second, the measure-theoretic argument involving the Calderón-Zygmund cube decomposition must be altered due to the presence of “bad” cubes of high ellipticity, and we actually use an alternative new idea based on the Besicovitch covering theorem.

The development is essentially self-contained and depends also on some novel uses of the area formula for Lipschitz functions, similar to the proof of the ABP inequality, and partially inspired by Savin [27, 28]. The main task of this section is to prove the following improvement of oscillation result.

Proposition 3.1 (Decay of oscillation). *There exists $\delta > 0$, depending only on d and Λ , such that if $0 < \mu < \frac{1}{2}$ and*

$$\int_{B_2 \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx < \delta,$$

then there exist constants $0 < \tau < 1$ depending only on d , Λ , and μ , such that for all $\alpha > 0$ and $u \in C(B_1)$ satisfying

$$\mathcal{P}_{\lambda(x), \Lambda}^+(D^2 u) \geq -\alpha \quad \text{and} \quad \mathcal{P}_{\lambda(x), \Lambda}^-(D^2 u) \leq \alpha \quad \text{in } B_1,$$

we have

$$\operatorname{osc}_{B_{1/4}} u \leq \tau \operatorname{osc}_{B_1} u + \alpha.$$

Proceeding with the proof of Proposition 3.1, we begin with three applications of area formula for Lipschitz functions (c.f. [16]), which asserts that

$$|f(E)| = \int_E |\det Df(x)| dx,$$

for all Lebesgue measurable sets $E \subseteq \mathbb{R}^d$ and injective Lipschitz maps $f : E \rightarrow \mathbb{R}^d$.

The first is the Alexandroff-Bakelman-Pucci (ABP) inequality. The version we give here is not new: it is actually a corollary to the proof of [6, Theorem 3.2]. We include a proof below both for completeness and in order to introduce the style of argument we use below, in a more complicated form, to obtain the barrier function. The argument here is much simpler than the one in [6], which is due to the observation that a semiconcave function is necessarily $C^{1,1}$ on the set where it can be touched from below by a plane.

Proposition 3.2 (ABP inequality). *If $u, f \in C(B_2)$ satisfy*

$$\begin{cases} \mathcal{P}_{\lambda(x), \Lambda}^+(D^2 u) \geq -f & \text{in } B_1 \\ u \geq 0 & \text{on } B_2 \setminus B_1, \end{cases}$$

then

$$u_-(0) \leq \left(\frac{1}{|B_1|} \int_{\{\Gamma_u=u\}} \lambda^{-d}(x) f_+^d(x) dx \right)^{1/d},$$

where

$$\Gamma_u(x) := \sup_{p \in \mathbb{R}^d} \inf_{y \in B_1} (p \cdot (x - y) - u_-(y))$$

is convex envelope of $-u_- := \min\{0, u\}$.

Proof. By approximating u by its infimal convolution, we may assume that u is a semiconcave (it is straightforward to see that the limsup of the contact sets for u_ε are contained in the contact set for u). Also assume that $a := u_-(0) > 0$, and observe that, for every $p \in B_a$, there exists $\bar{z}(p) \in B_2$ such that $u(\bar{z}(p)) < 0$ and the map $x \mapsto -u_-(x) - p \cdot x$ attains its infimum in B_2 at $x = \bar{z}(p)$. Note that we can arrange for $z : \bar{B}_a \rightarrow B_2$ to be Lebesgue measurable by choosing, say, lexicographically among the minimizers closest to the origin. Since u is semiconcave and can be touched from below by a plane on $A := \bar{z}(\bar{B}_a)$, it is $C^{1,1}$ on A . In particular, \bar{z} has a Lipschitz inverse $\bar{p}(z) := Du(z)$. Moreover, u is twice differentiable at Lebesgue almost every point of A by Rademacher's theorem (c.f. [16]) and the Lebesgue differentiation theorem. At such regular points $z \in A$, we have that $D^2u(z) \geq 0$, since u can be touched from below by a plane at z , and then the supersolution inequality gives

$$-f(z) \leq \mathcal{P}_{\lambda(z), \Lambda}^+(D^2u(z)) = -\lambda(z) \operatorname{tr}(D^2u(z)).$$

We deduce that, for almost every point z of A ,

$$(3.2) \quad 0 \leq D^2u(z) \leq \lambda^{-1}(z) f_+(z) I.$$

An application of the area formula yields

$$|B_1| a^d = |B_a| = \int_A |\det D\bar{p}(x)| dx \leq \int_A \lambda^{-d}(x) f_+^d(x) dx.$$

Since $A \subseteq \{\Gamma_u = u\}$, we obtain the proposition. \square

Using a more sophisticated version of the above argument, we next construct the barrier function, which below plays a critical role in the proof of Lemma 3.6 below, similar to that of the “bump” function in [6]. It is also needed in the next section in proof of Theorem 1 to verify that the limit function satisfies the Dirichlet boundary condition.

Lemma 3.3. *For each $0 < r < \frac{1}{2}$, there exists $\delta > 0$, depending only on d and Λ , such that if $0 < \mu < \frac{1}{2}$ and*

$$\int_{B_1 \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx < \delta r^d,$$

then there exists a constant $\beta > 0$, depending only on d , Λ , r , and μ , such that for each $u \in C(\bar{B}_1 \setminus B_r)$ satisfying

$$\begin{cases} \mathcal{P}_{\lambda(x), \Lambda}^+(D^2u) \geq -1 & \text{in } B_1 \setminus \bar{B}_r, \\ u \geq 0 & \text{on } \partial B_1, \\ u \geq \beta & \text{on } \partial B_r, \end{cases}$$

we have $u > 0$ on $B_{1-r} \setminus B_r$.

Proof. By approximating u by its infimal convolution, we may assume that u is semiconcave (alternatively, by the comparison principle we may replace u by the solution of the Dirichlet problem, which is C^2 by the Evans-Krylov theorem since $\mathcal{P}_{\lambda(x), \Lambda}^+$ is convex).

Define

$$\alpha := \frac{2(d-1)\Lambda}{\mu} - 1 \quad \text{and} \quad \beta := \left(\frac{4}{r}\right)^\alpha.$$

The main observation is that if u is negative somewhere in $B_{1-r} \setminus B_r$, then it can be touched from below by small translations of the singular function $\phi(x) := 2^\alpha |x|^{-\alpha}$. Indeed, suppose that $u(x_0) < 0$ for some $x_0 \in B_{1-r} \setminus B_r$. Then, for every $y \in B_{r/2}$,

$$(3.3) \quad \inf_{x \in \partial B_1} (u(x) - \phi(x-y)) \geq -2^\alpha \left|1 - \frac{r}{2}\right|^{-\alpha} > u(x_0) - \phi(x_0 - y)$$

and, by our choice of β ,

$$(3.4) \quad u(x) \geq \beta \geq \phi(x-y) \quad \text{for every } x \in \partial B_r.$$

Hence for every $y \in B_{r/2}$, the map $x \mapsto u(x) - \phi(x-y)$ attains its infimum at some point $\bar{z}(y) \in B_1 \setminus B_r$. We can easily arrange for the function $\bar{z} : B_{r/2} \rightarrow B_1 \setminus B_r$ to be Lebesgue measurable. To obtain a contradiction, we eventually apply the area formula to the inverse of \bar{z} . Most of the rest of the argument is concerned with showing that \bar{z} has an inverse \bar{y} and estimating $|\det D\bar{y}|$.

The differential inequality for u at $z = \bar{z}(y)$ yields

$$-1 \leq \mathcal{P}_{\lambda(z), \Lambda}^+(D^2\phi(z-y)) = \alpha(\alpha+1)2^\alpha |z-y|^{-(\alpha+2)} \left(\frac{\mu}{2} - \lambda(z)\right).$$

Using that $2^\alpha |z-y|^{-(\alpha+2)} \geq \frac{1}{4}$ and $\alpha(\alpha+1) \geq 8/\mu$, we obtain

$$(3.5) \quad \lambda(z) \leq \frac{\mu}{2} + \frac{4}{\alpha(\alpha+1)} < \mu.$$

We conclude that

$$(3.6) \quad A := \bar{z}(B_{r/2}) \subseteq B_1 \cap \{\lambda < \mu\}.$$

Since u is semiconcave and $|D^2\phi|$ is bounded in $\mathbb{R}^d \setminus B_{r/2}$, we see that u is $C^{1,1}$ on A . In particular, u is differentiable at each point of A and, by Rademacher's theorem and the Lebesgue differentiation theorem, twice differentiable at Lebesgue almost every point of A . For each $y \in B_{r/2}$,

$$Du(\bar{z}(y)) = D\phi(\bar{z}(y) - y) = -\alpha 2^\alpha |\bar{z}(y) - y|^{-(\alpha+2)} (\bar{z}(y) - y).$$

From this we see that \bar{z} has Lipschitz inverse $\bar{y} : A \rightarrow B_{r/2}$ given by

$$\bar{y}(z) := z + (\alpha 2^\alpha)^{-1} |z - y|^{\alpha+2} Du(z) = (\alpha 2^\alpha)^{1/(\alpha+1)} |Du(z)|^{-(\alpha+2)/(\alpha+1)} Du(z).$$

At points $z \in A$ where u is twice differentiable, we have

$$(3.7) \quad -\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} I \leq D^2\phi(z - \bar{y}(z)) \leq D^2u(z)$$

and, using the differential inequality,

$$-1 \leq \mathcal{P}_{\lambda(z), \Lambda}^+(D^2u(z)) \leq \Lambda \operatorname{tr}(D^2u(z))_- - \lambda(z) \operatorname{tr}(D^2u(z))_+.$$

A rearrangement of the later gives, in light of (3.7),

$$\operatorname{tr}(D^2u(z))_+ \leq \left(d\Lambda\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} + 1\right) \lambda^{-1}(z)$$

and from this we deduce that

$$(3.8) \quad D^2u(z) \leq C\lambda(z)^{-1}(\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} + 1)I,$$

where here and in the rest of the proof $C > 0$ depends only on d and Λ . Combining (3.7) and (3.8), we obtain, at each point $z \in A$ at which u is twice differentiable,

$$(3.9) \quad |D^2u(z)| \leq C\lambda^{-1}(z) \left(\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} + 1 \right).$$

Differentiating the formula for \bar{y} and using the expression for $Du(\bar{z}(y))$, we get

$$\begin{aligned} D\bar{y}(z) - I &= (\alpha 2^\alpha)^{-1} |z - \bar{y}(z)|^{\alpha+2} D^2u(z) + (\alpha 2^\alpha)^{-1} (\alpha + 2) |z - \bar{y}(z)|^\alpha (z - y) \otimes Du(z) \\ &= (\alpha 2^\alpha)^{-1} |z - \bar{y}(z)|^{\alpha+2} D^2u(z) + (\alpha + 2) |z - \bar{y}(z)|^{-2} (z - \bar{y}(z)) \otimes (z - \bar{y}(z)). \end{aligned}$$

Using (3.9) and that $2^{-\alpha} |z - \bar{y}(z)|^{\alpha+2} \leq 4$, we obtain

$$|D\bar{y}(z) - I| \leq C\lambda^{-1}(z) + (\alpha + 2).$$

Noticing that (3.6) implies that $\alpha + 2 \leq C\lambda^{-1}(z)$ for every $z \in A$, we at last deduce

$$(3.10) \quad |D\bar{y}(z) - I| \leq C\lambda^{-1}(z)$$

for Lebesgue almost every point of A .

We may finally apply the area formula, using (3.6), (3.10) and the hypothesis of the lemma to conclude that

$$\begin{aligned} (2^{-d}|B_1|) r^d &= |B_{r/2}| = \int_A |\det D\bar{y}(x)| dx \\ &\leq C \int_A \lambda^{-d}(x) dx \leq C \int_{B_1 \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx \leq C\delta r^d. \end{aligned}$$

We get a contradiction if $\delta > 0$ is sufficiently small, depending on d and Λ . \square

Remark 3.4. By an easy modification of the above proof, we obtain a slightly improved estimate for the barrier. Given $h > 0$ we can modify the choice of $\delta, \beta > 0$ and also select an $r' \in (r, 1)$ depending only on $d, \Lambda, \varepsilon, \mu$, such that the supersolution u satisfies $u \geq \beta - h$ on $B_{r'} \setminus B_r$. We use this observation to verify the Dirichlet boundary condition for the limit function in the proof of homogenization below.

The next lemma, which is inspired by [27, Lemma 2.1], follows from another application of the area formula and a (much easier) variation of the above argument. It asserts that, if a supersolution can be touched from below by sufficiently many translations of a fixed parabola, then the L^d norm of λ^{-1} on the set of points at which the touching occurs cannot be too small.

Lemma 3.5. *Suppose that $u \in C(B_1)$ satisfies*

$$\mathcal{P}_{\lambda(x), \Lambda}^+(D^2u) \geq -1 \quad \text{in } B_1,$$

and there is an open set $V \subseteq \mathbb{R}^d$ and a constant $a \geq 1$ such that the map $z \mapsto u(z) + \frac{a}{2}|z - y|^2$ attains its minimum in B_1 for all $y \in V$. Let $W \subseteq B_1$ denote the set of points at which, for some $y \in V$, this map attains its minimum. Then there exists a constant $\delta > 0$, depending only on d and Λ , such that

$$\int_W \lambda^{-d}(x) dx \geq \delta |V|.$$

Proof. By replacing u by $u + \alpha|x|^2$ and letting $\alpha \rightarrow 0$, we may suppose that, for some small $\eta > 0$, $W \subseteq B_{1-\eta}$ and, for every $y \in B_1$,

$$(3.11) \quad \min_{z \in \partial B_{1-\eta}} \left(u(z) + \frac{a}{2}|z - y|^2 \right) > \inf_{z \in W} \left(u(z) + \frac{a}{2}|z - y|^2 \right).$$

As in the previous two arguments, we may assume that u is semiconcave by infimal convolution approximation. Indeed, due to (3.11), the set V is essentially unchanged by the infimal convolution approximation, while the set W is unchanged or even smaller.

Select a Lebesgue-measurable function $\bar{z} : V \rightarrow B_1$ such that the map $z \mapsto u(z) + \frac{a}{2}|z - y|^2$ attains its infimum in B_1 at $z = \bar{z}(y)$. The function u is $C^{1,1}$ on $A := \bar{z}(V)$ and \bar{z} has a Lipschitz inverse \bar{y} given by

$$\bar{y}(z) := z + \frac{1}{a}Du(z).$$

By Rademacher's theorem and the Lebesgue differentiation theorem, u is twice differentiable at almost every point of A and $-aI \leq D^2u$ at such points z , where the differential inequality also gives

$$-1 \leq \mathcal{P}_{\lambda(z), \Lambda}^+(D^2u(z)) = \Lambda \operatorname{tr}(D^2u(z))_- - \lambda(x) \operatorname{tr}(D^2u(z))_+.$$

From this we obtain that, almost everywhere on A ,

$$(3.12) \quad -aI \leq D^2u(z) \leq I + ad\Lambda\lambda(z)^{-1}I \leq C\lambda^{-1}(z)(1+a)I.$$

An application of the area formula for Lipschitz functions yields

$$|V| = \int_A |\det D\bar{y}(x)| dx \leq C(1+a^{-1})^d \int_A \lambda^{-d}(x) dx,$$

from which we obtain the lemma, using that $a \geq 1$ and $A \subseteq W$. \square

We now give the proof of the decay of oscillation.

Proof of Proposition 3.1. Now suppose that $0 < \mu < \frac{1}{2}$ and

$$(3.13) \quad \int_{B_2 \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx < \delta := \frac{1}{6N_d} |B_{1/3}| \min \{8^{-d}\delta_1, 32^{-d}\delta_2\},$$

where δ_1 is from Lemma 3.3, δ_2 is from Lemma 3.5 and N_d is the constant from the Besicovitch covering theorem in dimension d .

We first make a reduction by noticing that, to obtain the proposition for $\tau := (1 - 1/k)$, it suffices to consider $v \in C(B_1)$ which satisfies

$$(3.14) \quad P_{\lambda(x), \Lambda}^+(D^2v) \geq -1 \quad \text{and} \quad P_{\lambda(x), \Lambda}^-(D^2v) \leq 1 \quad \text{in } B_1$$

and

$$(3.15) \quad \operatorname{osc}_{B_1} v \leq 1 + \operatorname{osc}_{B_{1/4}} v,$$

and to show that $\operatorname{osc}_{B_1} v < k$. Indeed, suppose u satisfies the hypotheses of the proposition and

$$\operatorname{osc}_{B_{1/4}} u > (1 - k^{-1}) \operatorname{osc}_{B_1} u + \alpha.$$

Then $\alpha < k^{-1} \operatorname{osc}_{B_1} u$, and so if we set $v := ku / \operatorname{osc}_{B_1} u$, then we see that v satisfies (3.14), (3.15), and $\operatorname{osc}_{B_1} v = k$.

Define, for every $\kappa > 0$,

$$A_\kappa := \left\{ x \in B_1 : \exists y \in B_1, v(x) + \frac{\kappa}{2}|x - y|^2 = \inf_{z \in B_1} \left(v(z) + \frac{\kappa}{2}|z - y|^2 \right) \right\}.$$

In other words, A_κ is the set of points in $B_{1/3}$ at which v can be touched from below by a paraboloid with Hessian $-\kappa I$ and vertex in B_1 . To prove the desired estimate on v , it is enough to show that

$$(3.16) \quad |A_\kappa \cap B_{1/3}| \geq \frac{2}{3}|B_{1/3}|$$

for some $\kappa > 0$ depending only on d , Λ and μ . Indeed, if we could show this, then using that $-v$ satisfies the same hypotheses as v and applying (3.16) to both functions, we find a point $x \in B_{1/3}$ which can be touched from above and below by parabolas with opening κ . That is, there exist $x \in B_{1/3}$ and $y_1, y_2 \in B_1$ such that, for all $z \in B_1$,

$$v(x) + \frac{\kappa}{2}|x - y_1|^2 - \frac{\kappa}{2}|z - y_1|^2 \leq v(z) \leq v(x) - \frac{\kappa}{2}|x - y_2|^2 + \frac{\kappa}{2}|z - y_2|^2.$$

This implies that $|v(x) - v(z)| \leq 8\kappa/9 \leq \kappa$ for all $z \in B_1$, and thus $\text{osc}_{B_1} v \leq 2\kappa$, which is the desired estimate.

In order to obtain (3.16) for some $\kappa > 0$ depending on the appropriate quantities, we observe first that (3.15) implies that $A_{144} \cap B_{1/3} \neq \emptyset$, since v can be touched from below in $B_{1/3}$ by the parabola $-144|x - y|^2$, where $y \in \overline{B}_{1/4}$ is such that $v(y) = \min_{\overline{B}_{1/4}} v$. We then repeatedly apply Lemma 3.6 below to obtain the desired result for $\kappa = 144\theta^n$, where $n := \lceil |B_1|/\eta \rceil$ and $\theta, \eta > 0$ are given in the statement of the lemma. The proof of Proposition 3.1 is now complete, pending the verification of Lemma 3.6. \square

To complete the proof of Proposition 3.1 we must prove the following lemma. In the classical regularity theory, a result like this typically relies on the Calderón-Zygmund cube decomposition (as in the proof of (4.12) in [6]). Since we did not see how to adapt it, we instead use an alternative measure-theoretic tool, the Besicovitch covering theorem¹. The argument also relies on Lemmas 3.3 and 3.5.

Lemma 3.6. *Let μ , v and A_κ be as in the proof of Proposition 3.1. There exist constant $\theta > 1$ and $\eta > 0$, depending only on d , Λ and μ , such that if $\kappa \geq 1$, $A_\kappa \cap B_{1/3} \neq \emptyset$ and $|A_\kappa \cap B_{1/3}| < \frac{2}{3}|B_{1/3}|$, then $|A_{\theta\kappa}| \geq |A_\kappa| + \eta$.*

Proof. Consider the collection \mathcal{B} of balls $B_r(x) \subseteq B_2$ such that $B_{r/2}(x) \subseteq B_1 \setminus A_\kappa$ and $\partial B_{r/2}(x) \cap A_\kappa \neq \emptyset$. Note that since $A_\kappa \cap B_{1/3} \neq \emptyset$ and A_κ is closed, every point of $B_{1/3} \setminus A_\kappa$ is the center of some ball in \mathcal{B} . According to the Besicovitch covering theorem, we may select a countable subcollection $\{B_{r_k}(x_k)\}_{k \in \mathbb{N}} \subseteq \mathcal{B}$ that covers $B_{1/3} \setminus A$ and such that each point $x \in B_2$ belongs to at most N_d balls.

We say that the ball $B_{r_k}(x_k)$ is *good* if

$$\int_{B_{r_k}(x_k) \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx < |B_{r_k}(x)| \min\{8^{-d}\delta_1, 32^{-d}\delta_2\}$$

¹Luis Silvestre has since pointed out to us that the Calderón-Zygmund decomposition argument in [6] may indeed be suitably modified to prove Lemma 3.6 and that, alternatively, the Vitali covering theorem may be used instead of the Besicovitch covering theorem with only small modifications to the argument we give below.

and set $G := \{k \in \mathbb{N} : B_{r_k}(x_k) \text{ is good}\}$, where $\delta_1, \delta_2 > 0$ are as in the proof of Proposition 3.1. We claim that at least half of the measure of $B_{1/3} \setminus A_\kappa$ consists of points which belong to good balls, i.e.,

$$(3.17) \quad \left| \bigcup_{k \in G} B_{r_k}(x_k) \right| \geq \frac{1}{2} |B_{1/3} \setminus A_\kappa| \geq \frac{1}{6} |B_{1/3}|.$$

Indeed, if (3.17) were false, then $\sum_{k \notin G} |B_{r_k}| \geq \frac{1}{2} |B_{1/3} \setminus A_\kappa| \geq \frac{1}{6} |B_{1/3}|$ and so

$$\begin{aligned} \int_{B_2 \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx &\geq \frac{1}{N_d} \sum_{k \notin G} \int_{B_{r_k}(x_k) \cap \{\lambda < \mu\}} \lambda^{-d}(x) dx \\ &\geq \frac{1}{N_d} \min \{8^{-d}\delta_1, 32^{-d}\delta_2\} \sum_{k \notin G} |B_{r_k}| \geq \frac{1}{6N_d} \min |B_{1/3}| \{8^{-d}\delta_1, 32^{-d}\delta_2\}, \end{aligned}$$

which contradicts (3.13). Therefore, in light of the Besicovitch covering, it is enough show that $|B_{r_k}(x_k) \cap A_{\theta\kappa}| \geq \eta |B_{r_k}(x_k)|$ for each good ball $B_{r_k}(x_k)$ and some constants $\theta, \eta > 0$ depending only on d, Λ , and μ .

Fix a good ball $B_r(x) := B_{r_k}(x_k)$ and choose $z_1 \in \partial B_{r/2}(x) \cap A_\kappa$. By the definition of A_κ , there exists $y_1 \in B_1$ such that

$$(3.18) \quad v(z_1) + \frac{\kappa}{2} |z_1 - y_1|^2 = \inf_{z \in B_1} \left(v(z) + \frac{\kappa}{2} |z - y_1|^2 \right).$$

Applying (a properly scaled) Lemma 3.3, since $B_r(x)$ is good there exists $\beta > 1$, depending only on d, Λ and μ , such that the solution w of the Dirichlet problem

$$\begin{cases} \mathcal{P}_{\lambda(x), \Lambda}^+(D^2 w) = -1 & \text{in } B_r(x) \setminus \overline{B}_{r/8}(x), \\ w = 0 & \text{on } \partial B_r(x), \\ w = \beta r^2 & \text{on } \partial B_{r/8}(x), \end{cases}$$

satisfies $w > 0$ in $\overline{B}_{r/2}(x) \setminus B_{r/8}(x)$. Clearly $w \leq \beta r^2$ in $B_r \setminus B_{r/8}(x)$ by the maximum principle. Observe that the function

$$\varphi(z) := (d\Lambda\kappa + 2)w - \frac{\kappa}{2} |z - y_1|^2,$$

satisfies

$$(3.19) \quad \mathcal{P}_{\lambda(x), \Lambda}^+(D^2 \varphi) \leq -2 \quad \text{in } B_r(x) \setminus B_{r/8}(x).$$

The comparison principle implies that the map $z \mapsto v(z) - \varphi(z)$ attains its infimum in $B_r(x) \setminus B_{r/8}(x)$ at some point $z = z_2 \in \partial B_r(x) \cup \partial B_{r/8}(x)$. However, $z_2 \in \partial B_r(x)$ is impossible due to (3.18) and the fact that $w \equiv 0$ on $\partial B_r(x)$ and $w(z_1) > 0$. Hence $z_2 \in \partial B_{r/8}(x)$, and so $\varphi(z_2) = -\frac{\kappa}{2} |z_2 - y_1|^2 + (d\Lambda\kappa + 2)\beta r^2$. We therefore obtain, using that $w > 0$ in $B_{r/2}(x) \setminus B_{r/8}(x)$,

$$\min_{z \in B_{r/2} \setminus B_{r/8}} \left(v(z) + \frac{\kappa}{2} |z - y_1|^2 \right) \geq v(z_2) + \frac{\kappa}{2} |z_2 - y_1|^2 - (d\Lambda + 2/\kappa) \kappa \beta r^2.$$

Using (3.18), $z_1 \in B_{r/2}(x)$ and $\kappa \geq 1$, we obtain

$$(3.20) \quad \inf_{z \in B_1} \left(v(z) + \frac{\kappa}{2} |z - y_1|^2 \right) \geq v(z_2) + \frac{\kappa}{2} |z_2 - y_1|^2 - (d\Lambda + 2) \kappa \beta r^2.$$

It follows that, if we set $\gamma := 16\beta(d\lambda + 2) + 1$, then for every $y_2 \in B_{r/8}(x)$, the function

$$\psi(z) := v(z) + \frac{\kappa}{2}|z - y_1|^2 + \frac{\gamma\kappa}{2}|z - y_2|^2$$

satisfies $\psi(z_2) < \min_{B_1 \setminus B_{r/2}(x)} \psi$ and therefore must attain its infimum in B_1 in $B_{r/2}(x)$. Consider the function $\bar{z}: B_{r/8}(x) \rightarrow B_1$ given by $\bar{z}(y) = (y_1 + \gamma y)/(1 + \gamma)$. Observe that for some constant $a \in \mathbb{R}$,

$$\frac{\kappa}{2}|z - y_1|^2 + \frac{\gamma\kappa}{2}|z - y_2|^2 = \frac{(\gamma + 1)\kappa}{2}|z - \bar{z}(y_2)|^2 + a \quad \text{for all } z \in \mathbb{R}^d.$$

It follows that the map $z \mapsto v(z) + \frac{\gamma\kappa}{2}|z - \bar{z}(y_2)|^2$ attains its infimum in B_1 at some point of $B_{r/2}(x)$. Since $\gamma \geq 1$ and thus $\gamma/(\gamma + 1) \geq \frac{1}{2}$, we deduce that

$$\bar{z}(B_{r/8}) \geq 2^{-d}|B_{r/8}(x)|.$$

Applying Lemma 3.5 and using that $B_r(x)$ is a good ball, we obtain

$$\begin{aligned} 2^{-d}\delta_2|B_{r/8}(x)| &\leq \int_{B_{r/2}(x) \cap A_{(\gamma+1)\kappa}} \lambda^{-d}(x) dx \\ &\leq 32^{-d}\delta_2|B_r(x)| + \mu^{-d}|B_{r/2}(x) \cap A_{(\gamma+1)\kappa}|, \end{aligned}$$

which implies $|B_{r/2}(x) \cap A_{(\gamma+1)\kappa}| \geq \mu^d(1 - 2^{-d})\delta_2|B_{r/8}(x)|$, as desired. \square

4. HOMOGENIZATION

The proof of homogenization follows the approach of [8], although we have re-organized the argument for clarity and simplicity as well as to accommodate the modifications required to handle the non-uniformly elliptic case. The reader interested only in the uniformly elliptic case can find here a concise proof of the result of [8]. The strategy relies on an application of the subadditive ergodic theorem to a certain quantity involving the obstacle problem. The proof has three steps:

- (1) Identifying \bar{F} : by applying the subadditive ergodic theorem to the Lebesgue measure of the contact set of a certain obstacle problem, we build the effective operator \bar{F} .
- (2) Building approximate correctors: with the help of the effective regularity results, we compare the solutions of the obstacle problem to the solution of the Dirichlet problem with zero boundary conditions and show that the latter act as approximate correctors.
- (3) Proving convergence: using the approximate correctors, the classical perturbed test function method allows us to conclude.

Step one: identifying \bar{F} via the obstacle problem. Following [8], we introduce, for each bounded Lipschitz domain $V \in \mathcal{L}$, the obstacle problem (with the zero function as the obstacle):

$$(4.1) \quad \begin{cases} \min \{F(D^2w, y, \omega), w\} = 0 & \text{in } V, \\ w = 0 & \text{on } \partial V, \end{cases}$$

Some important properties of (4.1) are reviewed in Appendix A. It is well-known that (4.1) has a unique viscosity solution, which we denote by $w = w(y, \omega; V, F)$. We often write $w = w(y, \omega; V)$ or simply $w = w(y, \omega)$ if we do not wish to display the dependence on F or V .

The set $\mathcal{C}(V, \omega) := \{y \in V : w(y, \omega; V) = 0\}$ of points where w touches the obstacle is called the *contact set*. We write $\mathcal{C}(V, \omega; F)$ if we wish to display the dependence on F . The Lebesgue measure of this set is an important quantity, and we denote it by

$$(4.2) \quad m(V, \omega) := |\mathcal{C}(V, \omega)|.$$

We check that m satisfies the hypotheses of the subadditive ergodic theorem (Proposition 2.2). First we observe from the monotonicity of the obstacle problem, see (A.4), that for all $V, W \in \mathcal{L}$ and $\omega \in \Omega$,

$$(4.3) \quad V \subseteq W \quad \text{implies that} \quad \mathcal{C}(W, \omega) \cap V \subseteq \mathcal{C}(V, \omega).$$

Immediate from (4.3) is the subadditivity of m . That is, for all $V, V_1, \dots, V_k \in \mathcal{L}$ such that $\cup_{j=1}^k V_j \subseteq V$, the sets V_1, \dots, V_k are pairwise disjoint and $|V \setminus \cup_{j=1}^k V_j| = 0$, we have

$$(4.4) \quad m(V, \omega) \leq \sum_{j=1}^k m(V_j, \omega).$$

According to (F1), m is stationary, that is,

$$m(V, \tau_y \omega) = m(y + V, \omega)$$

for every $y \in \mathbb{R}^d$ and $V \in \mathcal{L}$. We may easily extend m to \mathcal{U}_0 by defining, for every $A \in \mathcal{U}_0$,

$$\tilde{m}(A, \omega) := \inf \{m(V, \omega) : V \in \mathcal{L} \text{ and } A \subseteq V\}.$$

This extension agrees with m on \mathcal{L} by (4.3) and it is easy to show that the subadditivity and stationarity properties are preserved.

We now obtain:

Lemma 4.1. *There exists an event $\Omega_2 \in \mathcal{F}$ of full probability and a deterministic constant $\overline{m} \in \mathbb{R}$ such that, for every $\omega \in \Omega_2$ and Lipschitz domain $V \subseteq \mathbb{R}^d$,*

$$(4.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t^d} m(tV, \omega) = \overline{m} |V|.$$

Proof. In light of the remarks preceding the statement, the lemma follows from the subadditive ergodic theorem (Proposition 2.2). \square

For clarity we write $\overline{m} = \overline{m}(F)$ to display the dependence of \overline{m} in Lemma 4.1 on the nonlinear operator F .

We are now ready to define the effective nonlinearity:

$$(4.6) \quad \overline{F}(0) := \sup \{\alpha \in \mathbb{R} : \overline{m}(F - \alpha) > 0\}.$$

We extend this definition to all matrices in the obvious way. For each $N \in \mathbb{S}^d$ we denote F_N by

$$F_N(M, y, \omega) := F(M + N, y, \omega)$$

and then we set, for each $M \in \mathbb{S}^d$,

$$\overline{F}(M) := \overline{F}_M(0).$$

To check that \overline{F} is well-defined and finite, we first observe that, by (A.2) and (A.6),

$$\inf_{y \in V} F(0, y, \omega) \geq 0 \quad \text{implies that} \quad \mathcal{C}(V, \omega) = V$$

and

$$\sup_{y \in V} F(0, y, \omega) < 0 \quad \text{implies that} \quad \mathcal{C}(V, \omega) = \emptyset.$$

Using (F3) and the remarks in Section 2, it follows from these that

$$(4.7) \quad \operatorname{ess\,inf}_{\omega \in \Omega} F(M, 0, \omega) \leq \overline{F}(M) \leq \operatorname{ess\,sup}_{\omega \in \Omega} F(M, 0, \omega).$$

The monotonicity of the obstacle problem implies that $\alpha \mapsto \overline{m}(F - \alpha)$ is a decreasing function and thus $\overline{m}(F - \alpha) > 0$ for $\alpha < \overline{F}(0)$ and $\overline{m}(F - \alpha) = 0$ for $\alpha > \overline{F}(0)$.

It is immediate from the comparison principle for the obstacle problem that, if F_1 and F_2 are two operators satisfying our hypotheses, then

$$(4.8) \quad \sup_{M \in \mathbb{S}^d} \operatorname{ess\,sup}_{\omega \in \Omega} (F_1(M, 0, \omega) - F_2(M, 0, \omega)) \leq 0 \quad \text{implies} \quad \overline{F}_1 \leq \overline{F}_2.$$

It is even more obvious that adding constants commutes with the operation $F \mapsto \overline{F}$. From these facts a number of properties of \overline{F} are immediate, the ones inherited from uniform properties of F . A few of these are summarized in the following lemma.

Lemma 4.2. *For every $M, N \in \mathbb{S}^d$ such that $M \leq N$, we have*

$$(4.9) \quad 0 \leq \overline{F}(M) - \overline{F}(N) \leq \Lambda \operatorname{tr}(N - M).$$

Moreover, if $M \mapsto F(M, 0, \omega)$ is positively homogeneous of order one, odd, or linear, then \overline{F} possesses the same property.

Proof. Each of the properties are proved using the comments before the statement of the proposition. To prove (4.9), we simply observe that, according to (F1), for all $(Y, y, \omega) \in \mathbb{S}^d \times \mathbb{R}^d \times \Omega$,

$$(4.10) \quad F(M + Y, y, \omega) \leq F(N + Y, y, \omega) + \Lambda \operatorname{tr}(N - M)$$

and then apply (4.8). It is obvious that \overline{F} inherits the properties of positive homogeneity and oddness from F , and linearity follows from these. \square

Observe that (4.9) asserts that \overline{F} is degenerate elliptic. If F were uniformly elliptic, i.e., $\lambda^{-1} \in L^\infty(\Omega)$, then it follows from an argument nearly identical to the one for (4.9) that \overline{F} is uniformly elliptic. In our more general situation, i.e., $\lambda^{-1} \in L^d(\Omega)$, the operator \overline{F} is uniformly elliptic as well, but the proof is more complicated. We postpone it until the next subsection, since it is more convenient to deduce it as a consequence of Proposition 4.4.

We next show that, in large domains, the contact set has nearly constant density.

Lemma 4.3. *For every $\omega \in \Omega_2$ and $V, W \in \mathcal{L}$ with $\overline{W} \subseteq V$,*

$$(4.11) \quad \lim_{t \rightarrow \infty} \frac{|\mathcal{C}(tV, \omega) \cap tW|}{|tW|} = \overline{m}.$$

Proof. Let $U := V \setminus W \in \mathcal{L}$ and fix $\omega \in \Omega_2$. Observe that (4.3) gives

$$(4.12) \quad \limsup_{t \rightarrow \infty} \frac{|\mathcal{C}(tV, \omega) \cap tW|}{|tW|} \leq \lim_{t \rightarrow \infty} \frac{|\mathcal{C}(tW, \omega)|}{|tW|} = \overline{m}$$

and, by the same argument,

$$\limsup_{t \rightarrow \infty} \frac{|\mathcal{C}(tV, \omega) \cap tU|}{|tU|} \leq \overline{m}.$$

Therefore,

$$(4.13) \quad \liminf_{t \rightarrow \infty} \frac{|\mathcal{C}(tV, \omega) \cap tW|}{|tW|} = \liminf_{t \rightarrow \infty} \frac{|\mathcal{C}(tV, \omega) \cap tV| - |\mathcal{C}(tV, \omega) \cap tU|}{|tW|} \\ \geq \left(\frac{|V|}{|W|} - \frac{|U|}{|W|} \right) \overline{m} = \overline{m}.$$

Combining (4.12) and (4.13) yields (4.11). \square

Step two: building approximate correctors. The next step in the proof of Theorem 1 is to show that, in the macroscopic limit, the obstacle problem controls the solution of the Dirichlet problem

$$(4.14) \quad \begin{cases} F(D^2v, y, \omega) = 0 & \text{in } V, \\ v = 0 & \text{on } \partial V. \end{cases}$$

As before, $V \in \mathcal{L}$ is a bounded Lipschitz domain and we write $v = v(y, \omega; V, F)$.

The following proposition is the focus of this subsection.

Proposition 4.4. *There exists an event $\Omega_3 \in \mathcal{F}$ of full probability such that, for every $\omega \in \Omega_3$, $M \in \mathbb{S}^d$ and $V \in \mathcal{L}$,*

$$(4.15) \quad \lim_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tV} |v(y, \omega; tV, F_M - \overline{F}(M))| = 0.$$

Before we give its proof, we remark that Proposition 4.4 is a special case of Theorem 1. We can see this by fixing $U \in \mathcal{L}$, defining

$$v^\varepsilon(x, \omega) := \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega; \frac{1}{\varepsilon}U, F - \overline{F}(0)\right)$$

and then checking that $v^\varepsilon(\cdot, \omega)$ is the unique solution of the boundary-value problem

$$(4.16) \quad \begin{cases} F\left(D^2v^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = \overline{F}(0) & \text{in } U, \\ v^\varepsilon = 0 & \text{on } \partial U. \end{cases}$$

The conclusion of Proposition 4.4 then asserts that

$$(4.17) \quad v^\varepsilon \rightarrow 0 \quad \text{uniformly in } U \text{ as } \varepsilon \rightarrow 0,$$

which is consistent with Theorem 1 since the zero function $v \equiv 0$ is obviously the unique solution

$$(4.18) \quad \begin{cases} \overline{F}(D^2v) = \overline{F}(0) & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases}$$

As we show in the next subsection, Proposition 4.4 actually implies Theorem 1. This is because, for large $R > 0$, the function $\xi(y) := v(y, \omega; B_R, F_M - \overline{F}(M))$ is an “approximate corrector” in B_R in the sense that it satisfies the equation

$$(4.19) \quad F(M + D^2\xi, y, \omega) = \overline{F}(M) \quad \text{in } B_R,$$

and is “strictly subquadratic at infinity” (i.e., satisfies (4.15)). This is precisely what is needed to implement the perturbed test function method.

Proof of Proposition 4.4. According to the ergodic theorem, there exists an event $\Omega_4 \in \mathcal{F}$ of full probability such that, for every $\omega \in \Omega_4$, $V \in \mathcal{L}$ and rational $q \in \mathbb{Q}$ with $q > 0$,

$$(4.20) \quad \lim_{t \rightarrow \infty} \int_{tV} \lambda^{-d}(y, \omega) dy = \mathbb{E}[\lambda^{-d}]$$

and

$$(4.21) \quad \lim_{t \rightarrow \infty} \int_{tV} \lambda^{-d}(y, \omega) \chi_{\{\lambda < q\}}(y) dy = \mathbb{E}[\lambda^{-d} \mathbf{1}_{\{\lambda < q\}}].$$

Note that, according to the ABP inequality (Proposition 3.2, properly scaled), for every $\omega \in \Omega_4$,

$$(4.22) \quad \lim_{\alpha \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tV} \frac{1}{R^2} |v(y, \omega; tV, F) - v(y, \omega; tV, F + \alpha)| = 0.$$

We now define $\Omega_3 := \Omega_2 \cap \Omega_4$, where Ω_2 is given in the statement of Lemma 4.1.

We first show that, for all $\omega \in \Omega_3$, $V \in \mathcal{L}$, and $M \in \mathbb{S}^d$

$$(4.23) \quad \liminf_{t \rightarrow \infty} \frac{1}{t^2} \inf_{y \in tV} v(y, \omega; tV, F_M - \overline{F}(M)) \geq 0.$$

We may assume that $M = 0$ by replacing F with F_{-M} and that $\overline{F}(0) = 0$ by replacing F by $F - \overline{F}(0)$. By (4.22), we may also suppose that $\overline{m}(F) = 0$ by considering $F - \alpha$ for $\alpha > 0$ and then sending $\alpha \rightarrow 0$. Set

$$K := \operatorname{ess\,sup}_{\omega \in \Omega} (F(0, 0, \omega))_+ = \operatorname{ess\,sup}_{\omega \in \Omega} \sup_{y \in \mathbb{R}^d} (F(0, y, \omega))_+.$$

According to (4.15) and (A.5), for every $t > 0$, the function $u := w(\cdot, \omega; tV, F) - v(\cdot, \omega; tV, F)$ satisfies

$$\mathcal{P}_{\lambda(y, \omega), \Lambda}^-(D^2 u) \leq K \chi_{\mathcal{C}(tV, \omega)} \quad \text{in } tV$$

and $u = 0$ on $\partial(tV)$. Using that $w \geq 0$, the ABP inequality (Proposition 3.2, properly scaled) and (4.5), we obtain

$$(4.24) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tV} -v(y, \omega; tV, F) &\leq \limsup_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tV} u(y) \\ &\leq CK \limsup_{t \rightarrow \infty} \left(\int_{tV} \lambda^{-d}(y, \omega) \chi_{\mathcal{C}(tV, \omega)}(y) dy \right)^{1/d}. \end{aligned}$$

To estimate the integral on the right, we observe that, for each $k \in \mathbb{N}$,

$$\int_{tV} \lambda^{-d}(y, \omega) \chi_{\mathcal{C}(tV, \omega)}(y) dy \leq \left(k^d |m(tV, \omega)| + \int_{tV} \lambda^{-d}(y, \omega) \chi_{\{\lambda < 1/k\}}(y) dy \right).$$

Divide this by $|tV|$ and pass to the limit $t \rightarrow \infty$ using (4.21) to obtain

$$(4.25) \quad \limsup_{t \rightarrow \infty} \int_{tV} \lambda^{-d}(y, \omega) \chi_{\mathcal{C}(tV, \omega)}(y) dy \leq k^d \overline{m}(F) + \mathbb{E}[\lambda^{-d} \mathbf{1}_{\{\lambda < 1/k\}}].$$

Since $\overline{m}(F) = 0$, we may send $k \rightarrow \infty$ and combine the resulting expression with (4.24) to obtain (4.23).

To complete the proof, we show that, for every $\omega \in \Omega_3$,

$$(4.26) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tV} v(y, \omega; tV, F_M - \overline{F}(M)) \leq 0.$$

As above, we may suppose that $M = 0$ and $\overline{F}(0) = 0$. We may also assume that $\overline{m}(F) > 0$, by considering $F + \alpha$ for $\alpha > 0$ and then sending $\alpha \rightarrow 0$, using (4.22). Since $v \leq w$, it suffices for (4.26) to show that

$$(4.27) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tV} w(y, \omega; tV, F) = 0.$$

Furthermore, by the monotonicity of the obstacle problem it suffices to show that

$$(4.28) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^2} \sup_{y \in tB_R} w(y, \omega; tB_{2R}, F) = 0$$

where $R > 1$ is large enough that $V \subseteq B_R$. Fix $r > 0$ and observe that, by Lemma 4.3 and an easy covering argument using $\overline{m}(F) > 0$, there exists $T > 0$ sufficiently large such that, for every $t \geq T$ and $x \in B_R$, the function $w(\cdot, \omega; tB_{2R}, F)$ vanishes in $B(tx, tr)$. We therefore have, for every $t \geq T$ and $x \in B_R$,

$$(4.29) \quad \frac{1}{t^2} |w(tx, \omega; tB_{2R}; F)| \leq \operatorname{osc}_{B(tx, tr)} \frac{1}{t^2} w(\cdot, \omega; tB_{2R}; F).$$

We prove (4.28) by showing that the lim-sup of the right-side of (4.29), as $t \rightarrow \infty$, is $o(1)$ as $r \rightarrow 0$. For this we rely on Proposition 3.1.

Notice that (A.5), (4.20) and the ABP inequality (Proposition 3.2) yield, for $t > 0$ sufficiently large, the bound

$$(4.30) \quad \frac{1}{t^2} \sup_{tB_{2R}} |w(\cdot, \omega; tB_{2R}, F)| \leq CKR^2,$$

where C depends only on d, Λ and $\mathbb{E}[\lambda^{-d}]$. Select $\mu > 0$ such that

$$\mathbb{E}[\lambda^{-d} \mathbf{1}_{\{\lambda < \mu\}}] < 4^{-d} \delta,$$

where $\delta > 0$, $\delta \in \mathbb{Q}$ is as in Proposition 3.1. By (4.21), and making $T > 0$ larger, if necessary, we have that for all $t \geq T$, $r < r' < R$ and $x \in B_R$,

$$\int_{B_{tr'}(tx)} \lambda^{-d}(y, \omega) \chi_{\{\lambda < \mu\}} dy < \delta.$$

To see this, consider a finite covering $\{B_s(x_i)\}$ of B_R by balls of radius $s = 2^{-k}R$ for some $k \in \mathbb{N}$. According to (4.21), for sufficiently large t , the average of $\lambda^{-d}(y, \omega) \chi_{\{\lambda < \mu\}}$ in each of the balls $B_{2s}(x_i)$ will be less than $4^{-d} \delta$. But every ball $B_{r'}(x)$, with $s/2 \leq r' \leq s$ and $x \in B_R$, is contained in one of the balls $B_{2s}(x_i)$. Since $4r' \geq s$, this yields

$$(4.31) \quad \int_{B_{tr'}(tx)} \lambda^{-d}(y, \omega) \chi_{\{\lambda < \mu\}} dy \leq 4^d \int_{B_{2s}(x_i)} \lambda^{-d}(y, \omega) \chi_{\{\lambda < \mu\}} dy < \delta.$$

Repeating this covering argument for $k = 0, 1, 2, \dots, \lceil \log_2(R/r) \rceil$ and making $T > 0$ larger, if necessary, we obtain (4.31) for every $t \geq T$, $r \leq r' \leq R$ and $x \in B_R$.

Iterating Proposition 3.1, using (4.30) as well as (A.3) and (A.5), we obtain, for every $x \in B_R$ and $t \geq T$,

$$(4.32) \quad \operatorname{osc}_{B(tx, tr)} \frac{1}{t^2} w(\cdot, \omega; tB_{2R}; F) \leq Cr^\gamma$$

for some constants $\gamma > 0$ and $C > 0$ which may depend on $d, \Lambda, \mathbb{E}[\lambda^{-d}]$, μ, K and R , but do not depend on r . Combining this with (4.29) and sending $t \rightarrow \infty$ and then $r \rightarrow 0$, we obtain (4.27) and thus the proposition. \square

We conclude the second part by showing that \overline{F} is uniformly elliptic and giving an estimate of its ellipticity. The proof is based on Lemma 3.5 and Proposition 4.4.

Proposition 4.5. *There exists $\lambda_0 > 0$ depending only on d , Λ and $\mathbb{E}[\lambda^{-d}]$ such that, for all $M, N \in \mathbb{S}^d$,*

$$(4.33) \quad \mathcal{P}_{\lambda_0, \Lambda}^-(M - N) \leq \overline{F}(M) - \overline{F}(N) \leq \mathcal{P}_{\lambda_0, \Lambda}^+(M - N).$$

Proof. Select matrices $M, N \in \mathbb{S}^d$ such that $M \geq N$. Fix $\omega \in \Omega_3$ and define, for each $\varepsilon > 0$,

$$\begin{aligned} V_\varepsilon(x) := \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega; \frac{1}{\varepsilon} B_1, F_M - \overline{F}(M)\right) - \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega; \frac{1}{\varepsilon} B_1, F_N - \overline{F}(N)\right) \\ + \frac{1}{2} x \cdot (M - N)x. \end{aligned}$$

It is easy to check that V satisfies the inequality

$$(4.34) \quad \mathcal{P}_{\lambda(\frac{x}{\varepsilon}, \omega), \Lambda}^+(D^2 V_\varepsilon) \geq \overline{F}(M) - \overline{F}(N) \quad \text{in } B_1.$$

According to Proposition 4.4,

$$(4.35) \quad V_\varepsilon(x) \rightarrow \frac{1}{2} x \cdot (M - N)x \quad \text{as } \varepsilon \rightarrow 0 \quad \text{uniformly in } B_1.$$

Suppose that $M - N$ has a largest eigenvalue $a > 0$ with corresponding normalized eigenvector $\xi \in \mathbb{R}^d$, $|\xi| = 1$ so that

$$(4.36) \quad a\xi \otimes \xi \leq M - N \leq aI$$

Fix $\beta > 0$ and, for each $y \in \mathbb{R}^d$, denote by $\overline{z}(y) \in \mathbb{R}^d$ the (unique) point at which the map $x \mapsto \Phi(x, y) := \frac{1}{2} x \cdot (M - N)x + \beta|x - y|^2$ attains its (strict) global minimum on \mathbb{R}^d . The inequalities $\Phi(\overline{z}(y), y) \leq \Phi(0, y)$ and $\Phi(\overline{z}(y), y) \leq \Phi(y, y)$ imply, using (4.36), that

$$\frac{1}{2} a(\overline{z}(y) \cdot \xi)^2 + \beta|\overline{z}(y) - y|^2 \leq \min \left\{ \frac{1}{2} a|y|^2, \beta|y|^2 \right\}.$$

In particular, we notice that $|y| \leq 1/3$ implies that $|\overline{z}(y)| \leq 2/3$ and

$$(4.37) \quad |\overline{z}(y) \cdot \xi| \leq \left(\frac{2\beta}{9a} \right)^{1/2}.$$

Applying (4.35), we deduce that, for sufficiently small $\varepsilon > 0$ and every $y \in B_{1/3}$, the infimum in B_1 of the map $x \mapsto V_\varepsilon(x) + \beta|x - y|^2$ is attained in $B_{2/3}$ and any point z at which the minimum is attained satisfies

$$(4.38) \quad |z \cdot \xi| \leq \frac{1}{2} \left(\frac{\beta}{a} \right)^{1/2}.$$

Let $A := B_{2/3} \cap \{x : |x \cdot \xi| \leq (\beta/a)^{1/2}\}$ and observe that $|A| \leq C(\beta/a)^{1/2}$. If $\overline{F}(M) - \overline{F}(N) \geq -2\beta$, then we may apply Lemma 3.5, using (4.20), to obtain

$$c \leq |B_{1/3}| \leq C \limsup_{\varepsilon \rightarrow 0} \int_A \lambda \left(\frac{x}{\varepsilon}, \omega \right)^{-d} dx = C|A| \mathbb{E}[\lambda^{-d}] \leq C \left(\frac{\beta}{a} \right)^{1/2} \mathbb{E}[\lambda^{-d}].$$

This is impossible if $a \geq C\beta \mathbb{E}[\lambda^{-d}]^2$. Therefore, we may select $\lambda_0 > 0$ small enough, depending on the appropriate quantities, so that by fixing $\beta := \lambda_0 ad/2$, the previous argument yields, for all $M \geq N$,

$$\overline{F}(M) - \overline{F}(N) \leq -2\beta = -\lambda_0 ad = \mathcal{P}_{\lambda_0, \Lambda}^+(aI) \leq \mathcal{P}_{\lambda_0, \Lambda}^+(M - N).$$

Recalling (4.9), we also have, for every $M \geq N$,

$$\overline{F}(M) - \overline{F}(N) \geq -\Lambda \operatorname{tr}(N - M) = \mathcal{P}_{\lambda_0, \Lambda}^-(M - N).$$

We have verified (4.33) for all $M, N \in \mathbb{S}^d$ with $M \geq N$. To remove the latter restriction, fix $M, N \in \mathbb{S}^d$, write

$$\begin{aligned} \overline{F}(M) - \overline{F}(N) &= \overline{F}(M) - \overline{F}(M - (N - M)_-) \\ &\quad + \overline{F}(M - (N - M)_-) - \overline{F}(M - (N - M)_- + (N - M)_+) \end{aligned}$$

and observe by what we have shown above that

$$\overline{F}(M) - \overline{F}(N) \leq \mathcal{P}_{\lambda_0, \Lambda}^+((N - M)_-) - \mathcal{P}_{\lambda_0, \Lambda}^-((N - M)_+) = \mathcal{P}_{\lambda_0, \Lambda}^+(M - N).$$

Interchanging M and N yields the first inequality of (4.33) for all $M, N \in \mathbb{S}^d$. \square

Step three: concluding by the perturbed test function method. By adapting the classical perturbed test function method, first introduced in the context of periodic homogenization by Evans [15], we now complete the proof of Theorem 1. The test functions are perturbed by the approximate correctors constructed in Proposition 4.4. The argument we present here is similar in spirit to the one given in Section 4 of [8], although a bit less complicated.

Proof of Theorem 1. Fix a bounded Lipschitz domain $U \in \mathcal{L}$, $g \in C(\partial U)$ and an environment $\omega_0 \in \Omega_3$, where the event $\Omega_3 \in \mathcal{F}$ is given in the statement of Proposition 4.4.

We first argue that, for every $x \in U$,

$$(4.39) \quad \tilde{u}(x) := \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(x, \omega_0) \leq u(x).$$

To show (4.39), we begin by checking that $\tilde{u}(x) \leq g$ on ∂U . By approximation, we may assume that $g \equiv 0$ and that U is smooth (and in particular has the exterior ball condition). By dilation, we may also assume that $F(0, \cdot, \omega) \leq 1$ and that $U \subseteq B_{R/2}(0)$. Given $y \in \partial U$, we may select $B_r(x) \subseteq \mathbb{R}^d \setminus U$ such that $\overline{B_r}(x) \cap \overline{U} = \{y\}$. Given $h > 0$, we apply Lemma 3.3 with the modification in Remark 3.4. Using $\omega_0 \in \Omega_3$, we may select $\beta > 0$ and $r' \in (r, R - r)$ such that the solution $\varphi^\varepsilon \in C(\overline{B_R} \setminus B_r)$ of

$$\begin{cases} \mathcal{P}_{\lambda(x/\varepsilon, \omega_0), \Lambda}^-(D^2 \varphi^\varepsilon) = 1 & \text{in } B_R \setminus B_r \\ \varphi^\varepsilon = \beta & \text{on } \partial B_R \\ \varphi^\varepsilon = 0 & \text{on } \partial B_r, \end{cases}$$

satisfies $\limsup_{\varepsilon \rightarrow 0} \varphi^\varepsilon \leq h$ in $V \cap B_{r'}(x)$. Since $U \subseteq B_R(x)$ and $u^\varepsilon \leq 0$ on ∂U , the comparison principle implies that $u^\varepsilon \leq \varphi^\varepsilon$. It follows that

$$\limsup_{\varepsilon \rightarrow 0} \sup_{V \cap B_{r'-r}(y)} u^\varepsilon(\cdot, \omega_0) \leq h.$$

Since $h > 0$ was arbitrary, we conclude that $\tilde{u} \leq g$ on ∂U .

By the comparison principle, to prove (4.39) it suffices to check that the function $\tilde{u}(x) := \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(x, \omega_0)$ satisfies, in the viscosity sense,

$$(4.40) \quad F(D^2\tilde{u}) \leq 0 \quad \text{in } U.$$

To verify (4.40), we select a smooth test function $\phi \in C^2(U)$ and a point $x_0 \in U$ such that

$$x \mapsto (\tilde{u} - \phi)(x) \quad \text{has a strict local maximum at } x = x_0.$$

We must show that $\overline{F}(D^2\phi(x_0)) \leq 0$. Set $M := D^2\phi(x_0)$ and suppose on the contrary that $\theta := \overline{F}(M) > 0$.

Since the local maximum of $\tilde{u} - \phi$ at x_0 is strict, there exists $r_0 > 0$ such that $B_{r_0}(x_0) \subseteq U$ and, for every $0 < r \leq r_0$,

$$(4.41) \quad (\tilde{u} - \phi)(x_0) > \sup_{\partial B_r(x_0)} (\tilde{u} - \phi).$$

We next introduce the perturbed test function

$$\phi^\varepsilon(x) := \phi(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega_0; \frac{1}{\varepsilon} B_{r_0}(x_0), F_M - \overline{F}(M)\right).$$

We claim that, in some neighborhood of x_0 , ϕ^ε is a strict supersolution of the oscillatory equation at microscopic scale ε . More precisely, we will argue that, for some suitably small $0 < s < r_0$ to be selected below (and which may depend on ϕ),

$$(4.42) \quad F\left(D^2\phi^\varepsilon, \frac{x}{\varepsilon}, \omega_0\right) \geq \frac{1}{2}\theta \quad \text{in } B_s(x_0).$$

To check (4.42) we select a smooth test function $\psi \in C^2(B_s(x_0))$ and a point $x_1 \in B_s(x_0)$ such that

$$x \mapsto (\phi^\varepsilon - \psi)(x) \quad \text{has a local minimum at } x = x_1.$$

Using the definition of ϕ^ε and rescaling, we have

$$y \mapsto v\left(y, \omega_0; \frac{1}{\varepsilon} B_{r_0}(x_0), F_M - \overline{F}(M)\right) - \frac{1}{\varepsilon^2} (\psi(\varepsilon y) - \phi(\varepsilon y))$$

has a local minimum at $y = \frac{x_1}{\varepsilon}$.

Using the equation for v , we obtain

$$F\left(M + D^2\psi(x_1) - D^2\phi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) - \overline{F}(M) \geq 0.$$

Since $\phi \in C^2$, we may make $|M - D^2\phi(x_1)| = |D^2\phi(x_0) - D^2\phi(x_1)|$ as small as we like by taking $s > 0$ small enough. Thus, in light of (F2), we may fix $s > 0$ so that

$$\left| F\left(M + D^2\psi(x_1) - D^2\phi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) - F\left(D^2\psi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) \right| \leq \frac{1}{2}\theta.$$

The previous two inset inequalities and $\theta = \overline{F}(M)$ yield

$$(4.43) \quad F\left(D^2\psi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) \geq \frac{1}{2}\theta.$$

This completes the proof of (4.42).

An application of the comparison principle now yields

$$(4.44) \quad u^\varepsilon(x_0, \omega_0) - \phi^\varepsilon(x_0) \leq \sup_{B_s(x_0)} (u^\varepsilon(\cdot, \omega_0) - \phi^\varepsilon) = \sup_{\partial B_s(x_0)} (u^\varepsilon(\cdot, \omega_0) - \phi^\varepsilon).$$

Taking the limsup of both sides of (4.44) as $\varepsilon \rightarrow 0$ and applying Proposition 4.4, we obtain

$$\tilde{u}(x_0) - \phi(x_0) \leq \sup_{\partial B_s(x_0)} (\tilde{u} - \phi).$$

This contradicts (4.41) and completes the proof that $\overline{F}(M) \leq 0$, and hence of (4.40), and hence of (4.39).

It remains to show that, for every $x \in U$,

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(x, \omega_0) \geq u(x).$$

This is obtained by mimicking the argument above with only very obvious modifications, and so we omit the proof. \square

5. BREAKDOWN OF HOMOGENIZATION AND REGULARITY FOR $p < d$

In this section we show that the condition that the d th moment of λ^{-1} is finite is sharp for both the homogenization and regularity results. It was previously shown in [19] that homogenization may fail unless some moment condition is imposed. Here we construct a stationary-ergodic random environment $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ and an operator satisfying (F1), (F2) and (F3) as well as

$$(5.1) \quad \mathbb{E}[\lambda^{-p}] < +\infty \quad \text{for all } p < d,$$

but for which homogenization fails. The example is actually for the linear operator $F(M, y, \omega) := -\lambda(y, \omega) \operatorname{tr}(M)$, and we demonstrate the failure of homogenization (and an “effective” ABP inequality and hence effective regularity) by showing the solution u^ε of the Dirichlet problem

$$(5.2) \quad \begin{cases} -\lambda\left(\frac{x}{\varepsilon}, \omega\right) \Delta u^\varepsilon = 1 & \text{in } Q, \\ u^\varepsilon = 0 & \text{on } \partial Q, \end{cases}$$

satisfies $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(0, \omega) = +\infty$ almost surely, where $Q := [-1, 1]^d$. The uniformly continuous function $\lambda : \mathbb{R}^d \rightarrow (0, 1]$ we construct below satisfies a finite range dependence, so even this strongest possible ergodicity/mixing assumption cannot save homogenization without the moment condition (F4). The argument that $u^\varepsilon(0)$ blows up as $\varepsilon \rightarrow 0$ is a combination of the weak Harnack inequality and ideas from percolation.

The idea is to build “traps” where, from the probabilistic perspective, the corresponding diffusion will become stuck for long periods of time. We choose, for each positive integer k , a random arrangement $P_k(\omega) \subseteq \mathbb{R}^d$ of points so that $\lambda(y, \omega) \leq \lambda_k := 1/(k \log(2 + k))$ in each ball of radius 1 with center in $P_k(\omega)$. To be more precise, we select, for each $k \in \mathbb{N}$, a continuous function $\theta_k(y)$ on \mathbb{R}^d which is equal to λ_k in B_1 , takes the value 1 in $\mathbb{R}^d \setminus B_2$ and satisfies $\lambda_k \leq \theta_k(y) \leq 1$. Then we set

$$\lambda(y, \omega) := \inf_{k \in \mathbb{N}} \inf_{x \in P_k(\omega)} \theta_k(y - x).$$

The point configurations P_k are independent Poisson point processes (c.f. [13]), with intensity such that the expected number of elements of $P_k \cap V$ is equal to $a|V|k^{-1-d}$, where $a > 0$ is parameter we choose below. Since the series $\sum_{k=1}^{\infty} k^{-1-d}$

converges, it follows that the number of points of $\cup_{k=1}^{\infty} P_k$ is almost surely locally finite. Observe that the claimed moments of λ^{-1} are bounded: for all $p < d$,

$$\mathbb{E} [\lambda^{-p}] \leq 1 + Ca \sum_{k=1}^{\infty} \lambda_k^{-p} k^{-1-d} = 1 + Ca \sum_{k=1}^{\infty} k^{-1-d+p} \log^p(2+k) < \infty.$$

For each fixed $t > 1 + a|V|$, the probability of encountering at least one ball centered in tV in which $\lambda \leq \lambda_{[t]}$ is

$$(5.3) \quad \mathbb{P}[P_k \cap tV \neq \emptyset \text{ for some } k \geq t] \\ \geq 1 - \prod_{k=\lceil t \rceil}^{\infty} (1 - at^d |V| k^{-1-d}) \geq 1 - \exp(-a|V|).$$

We now choose the constant $a > 0$ to be large enough that the last quantity on the right, with $|V| = |Q|$, is larger than the critical percolation probability $p_c(d)$ in dimension d . It suffices to take $a := -2 \log(1 - p_c(d))$.

Having constructed the random field λ , we argue that homogenization fails for the equation (5.2) by demonstrating that, almost surely, the solution u^ε blows up at the origin as $\varepsilon \rightarrow 0$.

Suppose that $\varepsilon \ll 1$ and $x \in Q_{1-\varepsilon}$ and $x/\varepsilon \in \cup_{k \geq \lceil 1/\varepsilon \rceil} P_k$, so that x/ε is the center of a ball in which $\lambda \leq \lambda_{\lceil 1/\varepsilon \rceil}$. By a simple rescaling argument we see that

$$(5.4) \quad \inf_{B(x, \varepsilon/2)} u^\varepsilon(\cdot, \omega) - \inf_{\partial B(x, \varepsilon)} u^\varepsilon(\cdot, \omega) \geq c \lambda_{\lceil 1/\varepsilon \rceil}^{-1} \varepsilon^2 = c \varepsilon |\log \varepsilon|.$$

In light of (5.3), the probability of finding such a point x in each cube $y + \varepsilon Q$ is larger than $p_c(d)$. Since u^ε is superharmonic, $-\Delta u^\varepsilon \geq 0$ in B_1 , we can apply the weak Harnack inequality to deduce that, in such a cube we have

$$\inf_{y + \varepsilon Q/2} u^\varepsilon \geq \inf_{\partial(y + \varepsilon Q)} u^\varepsilon + c \varepsilon |\log \varepsilon|$$

Observe that the collection $\{y + \varepsilon Q : y \in (\varepsilon/2)\mathbb{Z}\}$ of cubes tiles \mathbb{R}^d . The shape theorem in first passage percolation (c.f. Kesten [22]) implies that there exists a constant $c_1 > 0$ such that, almost surely and for sufficiently small $\varepsilon > 0$, we can surround the origin by $c \lceil 1/\varepsilon \rceil$ many disjoint walls of these good cubes of size ε centered in $(\varepsilon/2)\mathbb{Z} \cap Q$. By a “wall” we mean a union of cubes of the form $y + \varepsilon Q$ with $y \in (\varepsilon/2)\mathbb{Z}$ such that the origin lies in a connected component of the complement. For each wall W , let $\partial W = \partial_i W \cup \partial_o W$ where $\partial_i W$ is the inner boundary (the boundary of the connected component of the complement of W containing the origin) and $\partial_o W$ is the outer boundary (the boundary of the unbounded connected component of W).

The construction above and another application of the weak Harnack inequality yield that, for each wall W ,

$$\inf_{\partial_i W} u^\varepsilon \geq \inf_{\partial_o W} u^\varepsilon + c \varepsilon |\log \varepsilon|.$$

Since there are $c \lceil 1/\varepsilon \rceil$ such walls between the origin and ∂Q , we deduce that

$$(5.5) \quad u^\varepsilon(0) \geq \inf_{\partial Q} u^\varepsilon + (c \lceil 1/\varepsilon \rceil) \varepsilon |\log \varepsilon| = c |\log \varepsilon| \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow 0.$$

This verifies that both homogenization and regularity break down for (5.2).

One wonders whether the failure of homogenization is simply due to the 1 on the right side of (5.2). Our second example, which is a variation of the first, considers

an equation with zero right side in dimension $d \geq 2$ and with ellipticity having finite p th moment for all $p \leq d - 1$.

Write $x = (x', x_d) \in \mathbb{R}^d$, with $x' \in \mathbb{R}^{d-1}$ and $Q := (-1, 1)^d$. Let $\lambda : \mathbb{R}^{d-1} \rightarrow (0, 1]$ be the random field constructed above, in one less dimension, and consider the Dirichlet problem

$$(5.6) \quad \begin{cases} -\lambda\left(\frac{x'}{\varepsilon}, \omega\right) \Delta' u^\varepsilon - u_{x_d x_d}^\varepsilon = 0 & \text{in } Q, \\ u^\varepsilon = g & \text{on } \partial Q, \end{cases}$$

where Δ' is the Laplacian with respect to the first $d - 1$ variables and $0 \leq g \leq 1$ is any continuous function on $\partial Q'$ which is equal to 1 on $\{x_{d+1} = \pm 1\}$ and vanishes in $\partial Q' \cap \{|x_{d+1}| \leq 1/2\}$. It is not difficult to show that the solution of (5.6) satisfies $u^\varepsilon \rightarrow 1$ locally uniformly in Q as $\varepsilon \rightarrow 0$, which rules out homogenization to any effective elliptic problem.

This confirms that homogenization may break down in dimensions larger than one for a linear operator even if the trace of its diffusion matrix is uniformly bounded below. In constructing the example, however, we destroyed the finite range dependence of the medium. We expect that this was necessary, since in the discrete setting Guo and Zeitouni [19] proved a quenched invariance principle for random walks in i.i.d random environments on the lattice, under the assumption that the random walker must move at each step (see also the interesting recent preprint of Berger and Deuschel [5] which extends this to the non-strictly elliptic setting). We conjecture that, in the continuum setting, homogenization holds if we impose both a finite range dependence and require the diffusion matrix to have trace uniformly bounded away from zero.

APPENDIX A. ELEMENTARY PROPERTIES OF THE OBSTACLE PROBLEM

For the convenience of the reader, we briefly review (and sketch the proofs of) some well-known properties of the obstacle problem

$$(A.1) \quad \begin{cases} \min \{F(D^2 w, y), w\} = 0 & \text{in } V, \\ w = 0 & \text{on } \partial V. \end{cases}$$

We have dropped the dependence of F on ω since the random environment plays no role here, and we furthermore assume that F is uniformly elliptic by the remarks preceding Theorem 1.

First of all, problem (A.1) has a unique solution $w \in C(\overline{V})$ which may be expressed as the least nonnegative supersolution of $F = 0$ in V :

$$(A.2) \quad w(x; V) = \inf \{u(x) : u \geq 0 \text{ in } V \text{ and } F(D^2 u, y) \geq 0 \text{ in } V\}.$$

This from the Perron method and the fact that the obstacle problem has a comparison principle. Immediate from this expression is that w is a global subsolution:

$$(A.3) \quad F(D^2 w, y) \geq 0 \quad \text{in } V,$$

as well as the monotonicity property:

$$(A.4) \quad V \subseteq W \quad \text{implies that} \quad w(\cdot; V) \leq w(\cdot; W) \quad \text{on } \overline{V}.$$

In order to use some regularity theory, we also need the fact that w satisfies

$$(A.5) \quad F(D^2w, y) \leq k\chi_{\{w=0\}} \quad \text{in } V \quad \text{where} \quad k := \sup_{y \in V} (F(0, y))_+.$$

This is typically handled by considering an approximate equation with a penalty term (the Levy-Stampacchia penalization method) whose solutions satisfy (A.5) and showing that their uniform limit is w . We instead opt for a more natural and simpler proof by showing that w is also given by the formula

$$(A.6) \quad w(x; V) = \sup \{u(x) : u \leq 0 \text{ on } \partial V \text{ and } F(D^2u, y) \leq k\chi_{\{u \leq 0\}} \text{ in } V\}.$$

It is clear that (A.6) implies (A.5). To check the former, we let \hat{w} denote the expression on the right side and observe that, since the zero function belongs to the admissible class by the definition of k , we have $\hat{w} \geq 0$. The Perron method implies that \hat{w} satisfies $F(D^2\hat{w}, y) = 0$ in $\{\hat{w} > 0\}$. Therefore \hat{w} is a solution of (4.1), and by uniqueness we deduce $w = \hat{w}$.

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